Exam Lie Groups in Physics

Date	November 4, 2015
Room	U 5173.0055
Time	9:00 - 12:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the three problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	8	2a)	10	3a)	7
1b)	5	2b)	10	3b)	10
1c)	5	2c)	8	3c)	5
1d)	7			3d)	5
1e)	5			3e)	5

$$\text{Result } = \frac{\sum \text{points}}{10} + 1$$

Problem 1

Consider the Lie group $U(n)$ of unitary $n \times n$ matrices and its subgroup $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

(a) Derive the dimensions of $U(n)$ and $SU(n)$.

(b) Show that $SU(n)$ forms an invariant subgroup of $U(n)$.

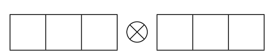
(c) Show that $U(n)/SU(n) \cong U(1)$.

(d) Determine the centers of $U(n)$ and $SU(n)$ (you may assume the defining reps are irreps).

(e) Give examples from physics where $SU(n)$ (for some $n \geq 2$) and $U(1)$ play a role as exact or approximate symmetries.

Problem 2

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(2)$ and $su(3)$. Indicate complex conjugate irreps whenever appropriate.

(c) Relate the decomposition for $su(2)$ to the corresponding case of addition of angular momenta in Quantum Mechanics.

Problem 3

Consider the group $SL(2,\mathsf{C})$ of 2×2 complex matrices with determinant equal to 1 and its relation to the Lorentz group.

(a) Explain (or demonstrate) why the 2-dimensional defining representation of $SL(2,\mathsf{C})$ is irreducible.

(b) Show that

$$\exp(-\chi\hat{n}\cdot\vec{\sigma}/2) = \mathbf{1}\cosh(\chi/2) - \hat{n}\cdot\vec{\sigma}\sinh(\chi/2),$$

for any unit vector $\hat{n}$ and real parameter χ. Recall that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ $(i = 1,2,3)$.

Now define $H(\chi,\hat{n}) \equiv \exp(-\chi\hat{n}\cdot\vec{\sigma}/2)$.

(c) Show that all $H(\chi,\hat{n})$ are elements of $SL(2,\mathsf{C})$, but only form a subset of $SL(2,\mathsf{C})$.

Next consider the homomorphism from $SL(2,\mathsf{C})$ to the Lorentz group $O(3,1)$: $A \mapsto L^{\mu}{}_{\nu} = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^{\mu}A\sigma_{\nu}A^{\dagger})$, where $\sigma^{\mu} = (\mathbf{1},\vec{\sigma})$ and $\tilde{\sigma}^{\mu} = (\mathbf{1},-\vec{\sigma})$, such that $\mathrm{Tr}(\sigma^{\mu}\tilde{\sigma}^{\nu}) = 2g^{\mu\nu}$.

(d) Explain why this homomorphism is neither 1-1 (injective) nor onto (surjective), when viewed as a mapping to the full Lorentz group $O(3,1)$.

(e) Explain which Lorentz transformations the elements $H \in SL(2,\mathsf{C})$ correspond to and whether transformations of this type form a (sub)group of the Lorentz transformations.